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Few Exact Solution of the Stokes' Problem with Slip at the Wall in Case of Suction/blowing

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Abstract

We provide an analytic solution for the problem of the unsteady, incompressible viscous fluid flow in the case of variable suction/blowing by applying similarity transform; the 'slip' condition at the boundary is considered. Several special cases with their solutions are given and are discussed graphically. It is observed that with an increase in suction the velocity increases and hence the boundary layer thickness decreases and with increase in blowing the velocity decreases and boundary layer thickness increases. It is also found that with an increase in slip parameter the velocity decreases.

Keywords: Variable suction; Slip flow; Similarity transform

1. Introduction

The known exact solutions of the Navier-Stokes equations are few in number, especially for two-directional unsteady flows and these become further rare when general variable suction velocity and free stream velocity was considered. This is due to the fact that solution can not be obtained without specifying some specific form of the variable suction velocity. Same problem occurs when the general plate velocity is taken at the boundary. As far as author is aware no attempt is made to discuss the physical results of the slip condition when the variable suction/blowing and the general free stream velocity are considered. However, a number of special cases are discussed by various authors, in which they considered constant suction/blowing, constant or oscillating free stream velocity and the "no-slip" condition (see for instance, the references (Stokes, 1901; Turbatu et al., 1998; Mohyuddin, 2006; Erdogan, 2000; Landau and Lifshitz, 1959)).

In the present analysis we consider a general free stream velocity at the boundary and general variable suction as a function of time. We also assume the slip between the velocity of the boundary and the plate. The relative velocity between u (0, t) and the plate is assumed to be proportional to the shear rate at the plate (Shiping, 2001; Khaled and Vafai, 2004). The solution of the problem is obtained by defining a new variable which transforms the problem into an ordinary differential equation. Eight special cases are discussed and graphs are plotted in each case to give the physical interpretation of the solution.

2. Problem formulation

Let us consider an unsteady flow of an incompressible viscous or Newtonian fluid. In the orthogonal coordinate system x, y, z, the behavior of such a fluid without body forces is usually described by the Navier-Stokes equations. The unsteady Navier-Stokes

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equations and the continuity equation are described by (Schlichting, 1968)

$$\rho \frac{d\mathbf{V}}{dt} = \mathrm{div}\mathbf{T}, \tag{1}$$

$$\mathrm{div}\mathbf{V} = 0 \tag{2}$$

where **V** is the velocity field, ρ is the density, *d/dt* is the mobile operator, *div* is the divergence and

$$\mathbf{T} = -p\mathbf{I} + \mu \mathbf{A}_1 \tag{3}$$

is the Cauchy stress tensor in which $-p\mathbf{I}$ is the indeterminate part of the stress (with pressure *p* and the identity tensor **I**), μ is the coefficient of viscosity and $\mathbf{A}_1 = \nabla\mathbf{V} + (\nabla\mathbf{V})^T$ is the first Rivlin-Ericksen tensor (Rivlin and Ericksen, 1955) or kinematic tensor in which $\nabla\mathbf{V}$ is the gradient of velocity vector and superscript *T* is the transpose.

We assume that an infinite permeable wall is aligned along the x−axis, and the y−axis is perpendicular to it, and the flow is planar. The fluid is bounded by the plate, which moves with general velocity *U(t)*. From the continuity equation the y−component of velocity is function of time only and equals *V (t)* (we take *V (t)* < 0 for suction and *V(t)* > 0 for blowing). The velocity field thus becomes

$$\mathbf{V}(y, t) = [u\,(y, t), \mathbf{V}\,(t), 0] \tag{4}$$

Using Eq. (4) in Eq. (1) we obtain

$$\frac{\partial u}{\partial t} + V(t)\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2}, \tag{5}$$

and

$$\rho \frac{d}{dt} V(t) = -\frac{\partial p}{\partial y} \tag{6}$$

Defining the modified pressure

$$\hat{p} = -\rho \frac{dV}{dt} y + p_0(t) \tag{7}$$

Equation (5) becomes

$$\frac{\partial u}{\partial t} + V(t)\frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}, \tag{8}$$

where $p_0\,(t)$ is the reference pressure, $\nu = \mu / \rho$ is the kinematic viscosity and u is the velocity in the x−direction. It is required to find the velocity distribution as a function of time and the space coordinate y.

For the boundary condition we assume the existence of slip between the velocity and the plate. The relative velocity between *u (0, t) and* the plate is assumed to be proportional to the shear rate at the plate (Khaled and Vafai, 2004)

$$u(0,t) - \beta \frac{\partial u(0,t)}{\partial y} = U(t) \tag{9}$$

$$u(y,t) \to 0 \text{ as } y \to \infty \tag{10}$$

where β is the slip parameter ($\beta = 0$ gives the usual no-slip condition). We have assumed that the fluid is bounded at infinity (Bird et al., 1960) that is

$$u(y,t) \to 0 \quad as \quad y \to \infty \,. \tag{11}$$

3. **Solution**

The partial differential equation described in equation (8) can be reduced to a linear ordinary differential equation by defining the similarity transformation as:

$$\varphi = \frac{y - \int V(t)dt}{2\sqrt{\nu t}} \tag{12}$$

After transforming Eq. (8) and the boundary conditions (9) and (10) we obtain the following system.

$$\frac{d^2 u}{d\varphi^2} + 2\varphi \frac{du}{d\varphi} = 0, \tag{13}$$

$$\left[u(\varphi) - \alpha \frac{du}{d\varphi} \right]_{\varphi=\lambda(t)} = \mathrm{U(t)}\,, \tag{14}$$

$$u \to 0 \quad \text{as} \quad \varphi \to \infty\,, \tag{15}$$

where prime denotes the differentiation with respect to variable η and

$$\alpha = \frac{\beta}{2\sqrt{\nu t}}, \lambda(t) = -\frac{\int V(t)dt}{2\sqrt{\nu t}}\,. \tag{16}$$

The general solution of Eq. (13) is

$$u(\varphi) = c_1 \int_0^{\varphi} e^{-\xi^2} d\xi + c_2 , \quad (17)$$

where c_1 and c_2 are integration constants. In order to satisfy the Eq. (15) we take $c_2 = -\frac{\sqrt{\pi}}{2} c_1$ to get

$$u(\varphi) = c_1 \left(erf(\varphi) - \frac{\sqrt{\pi}}{2} \right) = A(1 - erf(\varphi)), \quad (18)$$

where $erf(\varphi) = \frac{2}{\sqrt{\pi}} \int_0^{\varphi} e^{-\xi^2} d\xi$ is the error function.

In order to satisfy the boundary conditions (14) we require that

$$A = U(t) \left[erfc(\lambda(t) + \frac{2\alpha}{\sqrt{\pi}} e^{-\lambda(t)^2} \right]^{-1} . \quad (19)$$

Because the Eqs. (16) and (18) show that A is a function of time only. Since A is the integration constant, it should be independent of time which is possible only if

$$U(t) = (const) \left[erfc(\lambda(t) + \frac{2\alpha}{\sqrt{\pi}} e^{-\lambda(t)^2} \right]^{-1} . \quad (20)$$

Equation (20) is the relationship between *U (t)* and *V (t)* and at the same time it shows the consistency of *A* from equation (19) and it further shows that *A* is a fixed number.

Since from Eqs. (16), (17) and (20) we see that

$$u = u(y, t, U(t), V(t)) , \quad (21)$$

it is necessary to describe a function in the form of Eq. (20) and at the same time *U (t)* , (from Eq. (19)) , defines the value of *A* which significantly affects the value of *u*. Hence we conclude that the set of equations

$$p = p_0(t) - \rho \frac{dV(t)}{dt} y, \quad (22)$$

$$u = A \left[1 - erf \left(\frac{y - \int V(t) dt}{2\sqrt{\nu t}} \right) \right], \quad (23)$$

$$v = V(t), \quad (24)$$

$$U(t) = (const) \left[erfc(\lambda(t) + \frac{2\alpha}{\sqrt{\pi}} e^{-\lambda(t)^2} \right], \quad (25)$$

are the solution of the problem under consideration. In Eq. (25) *erf c (x) = 1 − erf (x)* is the complementary error function.

4. Special cases

In order to understand the physical aspects of solution given by Eqs. (23)~(25) we consider some special cases.

4.1 Case 1

We assume that the wall is rigid and there is 'no slip' at the wall that is, when *V (t) = 0 ,* $\beta = 0$ and $U(t) = U_0$ we get, from Eqs. (22)~(25), the following solution (Schlichting, 1968)

$$p = p_0(t), \quad (26)$$

$$u = U_0 \left[1 - erf \left(\frac{y}{2\sqrt{\nu t}} \right) \right] . \quad (27)$$

Equation (27) is the well known Stokes' first problem. Defining

$$f = \frac{u}{U_0}, \ \xi = \frac{U_0}{\nu} y , \ \tau = \frac{U_0^2}{\nu} t , \quad (28)$$

we get the non-dimensional form of Eq.(30) given as

$$f(\xi, \tau) = 1 - erf(\eta) , \quad (29)$$

where

$$\eta = \frac{\xi}{2\sqrt{\tau}} .$$

4.2 Case 2

In this case we assume that there is 'no slip' ($\beta = 0$) at the wall, $U(t) = U_0$ and the suction or blowing velocity is of the form $V(t) = \frac{k}{2} \sqrt{\frac{\nu}{t}}$. The solution is given as

$$p = p_0(t) + \frac{\rho k}{4} \sqrt{\frac{\nu}{t}} \frac{y}{t} , \quad (30)$$

$$u = \frac{1 - erf \left(\frac{y}{2\sqrt{\nu t}} - \frac{k}{2} \right)}{1 + erf \left(\frac{k}{2} \right)} . \quad (31)$$

The non-dimensional form of Eq. (31) is

$$f(\xi,\tau)=\frac{1-erf\left(\eta-\frac{k}{2}\right)}{1+erf\left(\frac{k}{2}\right)}. \qquad (32)$$

The velocity f against η for various values of k, suction (k = -2) and blowing (k = 2, 4), is plotted in Fig. 3.

4.3 Case 3

In this case we consider the slip ($\beta \neq 0$) at the wall, zero suction or blowing velocity $V(t) = 0$, and $U(t)=U_0$ and we get

$$p = p_0(t), \qquad (33)$$

$$u(y,t)=\frac{U_0\left[1-erf\left(\frac{y}{2\sqrt{\nu t}}\right)\right]}{1+\frac{2\alpha}{\sqrt{\pi}}}. \qquad (34)$$

The dimensionless form of Eq. (34) is

$$f(\xi,\tau)=\frac{1-erf(\eta)}{1+\frac{2\alpha}{\sqrt{\pi}}}. \qquad (35)$$

The velocity f against η for various values of α, the slip parameter, is plotted in Fig. 1.

4.4 Case 4

In this case we will take into account both the slip condition ($\beta \neq 0$) and the

suction or blowing velocity $\left(V(t)=\frac{k}{2}\sqrt{\frac{\nu}{t}}\right)$ and $U(t) = U_0$ and obtain

$$p = p_0(t)-\frac{\rho k}{4}\sqrt{\frac{\nu}{t}}\frac{y}{t}, \qquad (36)$$

$$u(y,t)=\frac{U_0\left[1-erf\left(\frac{y}{2\sqrt{\nu t}}-\frac{k}{2}\right)\right]}{1+erf\left(\frac{k}{2}\right)+\frac{2\alpha}{\sqrt{\pi}}e^{-\frac{k^2}{4}}}. \qquad (37)$$

The non-dimensional form of Eq. (37) is

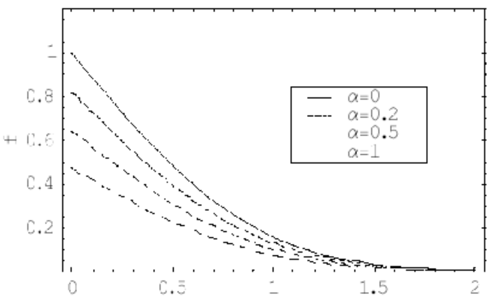

Fig. 1. The velocity f against η for different values of the slip parameter.

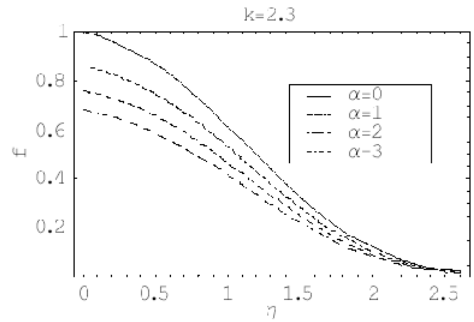

Fig. 2. The velocity f against η for different values of the slip parameter and $k = 2.3$.

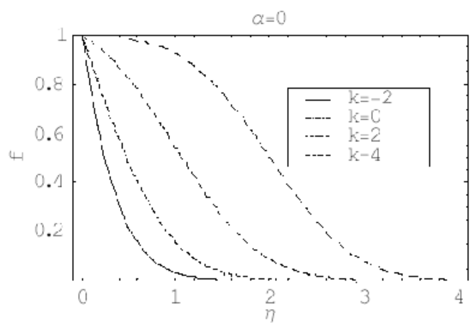

Fig. 3. The velocity f against η for different values of suction/blowing parameter, without slip.

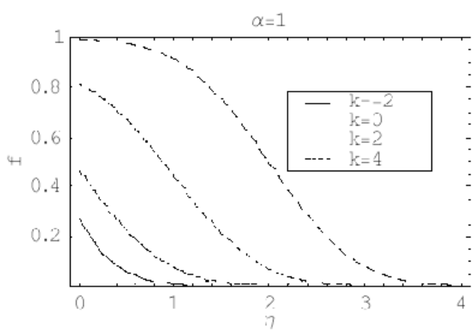

Fig. 4. The velocity f against η for different values of suction/blowing parameter, with slip.

$$f(\xi,\tau)=\frac{1-erf\left(\eta-\frac{k}{2}\right)}{1+erf\left(\frac{k}{2}\right)+\frac{2\alpha}{\sqrt{\pi}}e^{-\frac{k^2}{4}}}. \quad (38)$$

The velocity f against η for various values of slip parameter α and suction/blowing parameter k, suction (k = -2) and blowing (k = 2,4), are plotted in Figs. 2~4.

4.5 Case 5

In this case we assume that the suction/blowing velocity is linear in the variable t i.e., $V(t) = at$ and $U(t) = U_0$, $\beta \neq 0$, then the pressure field and velocity profile are given as

$$p = p_0(t) - a\rho y, \quad (39)$$

and

$$u(y,t)=\frac{U_0\left[1-erf\left(\frac{2y-at^2}{4\sqrt{\nu t}}\right)\right]}{1+erf\left(\frac{at^2}{4\sqrt{\nu t}}\right)+\frac{2\alpha}{\sqrt{\pi}}e^{-\frac{a^2t^3}{16\nu}}}. \quad (40)$$

The dimensionless form of Eq.(3.5.2) is

$$f(\xi,\tau)=\frac{1-erf\left(\eta-\frac{\bar{a}}{4}\tau^{\frac{3}{2}}\right)}{1+erf\left(\frac{\bar{a}}{4}\tau^{\frac{3}{2}}\right)+\frac{2\alpha}{\sqrt{\pi}}e^{-\frac{\bar{a}^2\tau^3}{16}}}, \quad (41)$$

where $\bar{a} = a\nu/U_0^3$. In the limiting case (when $\tau \to \infty$) the expression (41) exactly approaches to unity.

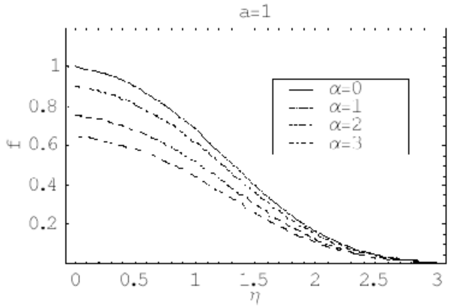

Fig. 5. The velocity f against η for different values of slip parameter and $a = 1$.

Figure 1 is plotted for velocity f against η when (α = 0, 0.2, 0.5,1) . It is seen that with the increase in slip parameter the velocity decreases. Figure 2 is plotted for velocity f against η when (α = 0, 1, 2,3 and k = 2.3) . It is clear from this figure that the velocity attains its maximum value (f = 1) and gradually decreases with the increase in α and becomes smooth at η = 2.6. Figure 3 is plotted for velocity f against η when (α = 0 and k = −2, 0, 2, 4) . We see that for different values of k velocity starts with its maximum value one and converges rapidly with the decrease of parameter k. Figure 4 is plotted for velocity f against η when (α = 1 and k = −2, 0, 2, 4) and it shows that velocity the decreases with the increase in slip parameter α . It is observed from the Figs. 3 and 4 that the parameter k shows the behavior of suction and blowing, that is, for k=2,4 it is the case of blowing for which the boundary layer thickness increases and for k=-2 it is the case of suction for which the boundary layer thickness decrease, whereas for k=0 there is no suction or blowing in the flow region. Figure 5 is plotted for velocity f against η when ($\bar{a}$ = 1, τ = 3 and α = 0, 1, 2, 3). It is observed that for $\tau \geq 7$, the velocity coincides with the free stream velocity. Similar observation is found when $\bar{a} \geq$ 3.7.

4.6 Case 6

Here we assume the exponential type suction/ blowing velocity i.e., $V(t) = U_0 e^{ct}$ and $U(t) = U_0$, $\beta \neq 0$. Expressions are written of the following form

$$p = p_0(t) - c\rho U_0 e^{ct} y, \quad (42)$$

and

$$u(y,t)=\frac{U_0\left[1-erf\left(\frac{y-\frac{U_0}{c}e^{ct}}{2\sqrt{\nu t}}\right)\right]}{1-erf\left(\frac{U_0e^{ct}}{2c\sqrt{\nu t}}\right)+\frac{2\alpha}{\sqrt{\pi}}e^{-\frac{U_0^2e^{2ct}}{4c^2\nu t}}}. \quad (43)$$

The non-dimensional form of Eq. (43) is

$$f(\xi,\tau)=\frac{1-erf\left(\eta-\frac{1}{2\bar{c}\sqrt{\tau}}e^{\bar{c}\tau}\right)}{1-erf\left(\frac{1}{2\bar{c}\sqrt{\tau}}e^{\bar{c}\tau}\right)+\frac{2\alpha}{\sqrt{\pi}}e^{-\frac{e^{2\bar{c}\tau}}{4\bar{c}^2\tau}}}, \quad (44)$$

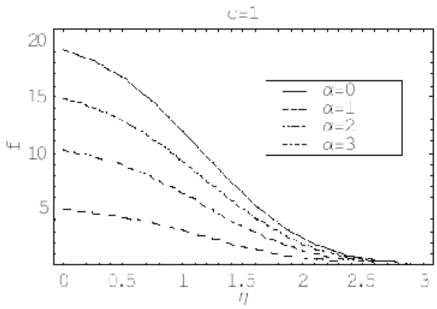

Fig. 6. The velocity f against η for different values of slip parameter and $c=1$.

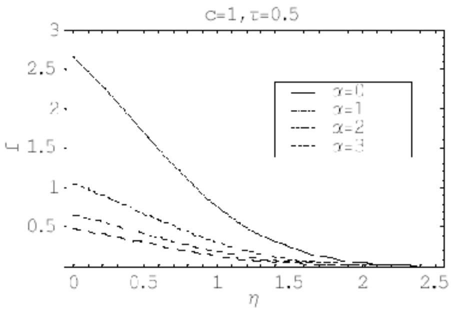

Fig. 7(a). The velocity f against η for different values of slip parameter and $c=1,\ \tau=0.5$.

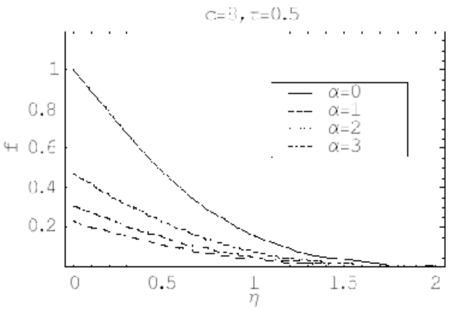

Fig. 7(b). The velocity f against η for different values of slip parameter and $c=8,\ \tau=0.5$.

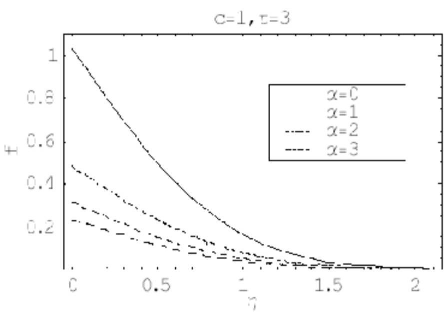

Fig. 7©. The velocity f against η for different values of slip parameter and $c=1,\ \tau=3$.

where $\mathrm{c} = cv/U_0^2$.

Figure 6 is plotted for velocity f against η when ($\bar{c}$ = 1, τ = 0.5 and α = 0, 1, 2, 3). In Fig. 6 the velocity decreases with increase in slip parameter and becomes uniform for $\bar{c}$ = 1 and $\eta \geq 3$.

4.7 Case 7

The results when V (t) = U_0e^{-ct}, U (t) = U_0, and $\beta \neq 0$ are

$$p = p_0(t) + c\rho U_0 e^{-ct} y\ , \tag{45}$$

$$u(y,t) = \frac{U_0\left[1 - erf\left(\dfrac{y + \dfrac{U_0}{c}e^{-ct}}{2\sqrt{vt}}\right)\right]}{1 - erf\left(\dfrac{U_0e^{-ct}}{2c\sqrt{vt}}\right) + \dfrac{2\alpha}{\sqrt{\pi}} e^{-\frac{U_0^2e^{-2ct}}{4c^2vt}}}\ . \tag{46}$$

The non-dimensional form of Eq. (46) is

$$f(\xi,\tau) = \frac{1 - erf\left(\eta - \dfrac{1}{2\bar{c}\sqrt{\tau}}e^{-\bar{c}\tau}\right)}{1 - erf\left(\dfrac{1}{2\bar{c}\sqrt{\tau}}e^{-\bar{c}\tau}\right) + \dfrac{2\alpha}{\sqrt{\pi}} e^{-\frac{e^{-2\bar{c}\tau}}{4\bar{c}^2\tau}}}\ . \tag{47}$$

Figure 7 is plotted for velocity f against η when ($\bar{c}$ = 1, τ = 0.5 and α = 0, 1, 2, 3), ($\bar{c}$ = 8, τ = 0.5 and α = 0, 1, 2, 3) and ($\bar{c}$ = 1, τ = 3 and α = 0, 1, 2, 3). In Fig. 7(a) the velocity decreases with increase in slip parameter and becomes uniform for $\bar{c}$ = 1 and $\eta \geq 2.5$. Comparing Fig. 7(b) with Fig. 7(b) we see that the velocity remains bounded in Fig. 7(b) whereas in Fig. 7(b) it was unbounded and decreases as we increase the parameter c (c ≥ 8) for τ = 0.5, the velocity becomes infinite at the plate. Similar results are obtained by comparing Fig. 7(c) with Fig. 7(c) for $\tau \geq 3$.

4.8 Case 8

In this case we consider oscillatory type suction/blowing velocity i.e., V (t) = U_0 (1 + ξe^{iwt}) and U (t) = U_0, $\beta \neq 0$, and get the following pressure and velocity fields

$$p = p_0(t) - i\rho\omega U_0 \varepsilon e^{i\omega t} y\ , \tag{48}$$

and

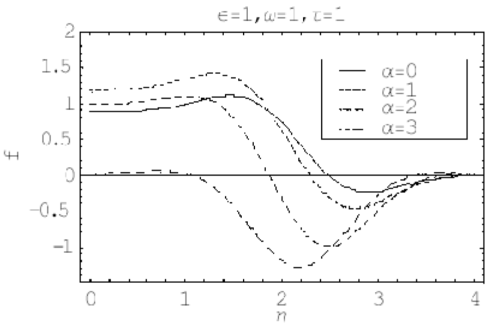

Fig. 8. The velocity f against η for different values of slip parameter and $\varepsilon = 1,\ \omega = 1,\ \tau = 3$.

$$u(y,t) = \frac{U_0\left[1 - erf\left(\dfrac{y - U_0 t + \dfrac{i\varepsilon U_0}{\omega} e^{i\omega t}}{2\sqrt{\nu t}}\right)\right]}{1 - erf\left(\dfrac{U_0 t + \dfrac{i\varepsilon U_0}{\omega} e^{i\omega t}}{2\sqrt{\nu t}}\right) + \dfrac{2\alpha}{\sqrt{\pi}} e^{-\frac{\left(U_0 t + \frac{i\varepsilon U_0}{\omega} e^{i\omega t}\right)^2}{4\nu t}}}, \tag{49}$$

$$f(\xi,\tau) = \frac{1 - erf\left(\eta - \dfrac{\tau^{\frac{1}{2}}}{2} + \dfrac{i\varepsilon}{2\bar{\omega}\sqrt{\tau}} e^{i\bar{\omega}\tau}\right)}{1 - erf\left(\dfrac{\tau^{\frac{1}{2}}}{2} + \dfrac{i\varepsilon}{2\bar{\omega}\sqrt{\tau}} e^{i\bar{\omega}\tau}\right) + \dfrac{2\alpha}{\sqrt{\pi}} e^{-\left(\frac{\tau^{\frac{1}{2}}}{2} + \frac{i\varepsilon}{2\bar{\omega}\sqrt{\tau}} e^{i\bar{\omega}\tau}\right)^2}}, \tag{50}$$

where $\bar{\omega} = \nu\omega / U_0^2$.

Figure 8 is plotted for velocity f against η when (ε = 1, $\bar{\omega}$ = 1, τ = 1 and α = 0, 1, 2, 3). We see that as the slip parameter increases the amplitude of oscillation also increases, thus velocity decreases and becomes uniform at $\eta = 4$.

5. Conclusion

In the present investigation we have solved the unsteady viscous two directional problem with the effects of the slip with the general time dependent velocity at the boundary. We find the general solution and then discuss it under eight different cases. From these cases we have observed the following:

With an increase in suction velocity the fluid velocity increases and with increase in blowing velocity the fluid velocity decreases, which is physically expected.

The boundary layer thickness decreases in the case of suction and increases for blowing (see the cases 2 and 4 for k=-2,0,2,4, where k=-2 is suction and k=2,4 is blowing).

With the increase in slip parameter the velocity decreases gradually.

It is found that the velocity effects remain prominent in the vicinity of the boundary and coincides with the free stream velocity far from the plate (as in case 6) i.e., $\tau \geq 3$.

The general solution reduces to Stokes first problem (Stokes, 1901) if we take V (t) = 0, β = 0 and U (t) = U_0.

If we take V (t) = $\dfrac{k}{2}\sqrt{\dfrac{\nu}{t}}$ the solution becomes consistent with (Jahnke et al., 1960).

If we choose α = 0 the solution matches with (Kozlov and Ptukha, 1981).

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